In order to assure a continued comparison between laboratories, we realize the necessity of adopting a traveling standard for the measure of the RF and MW electromagnetic-field intensities. Furthermore, it should be desirable to adopt a system (dummy plus sensor) for the reciprocal comparison of dosimetric measurements.

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# Electromagnetic Waves in Conical Waveguides with Elliptic Cross Section

S. BLUME AND B. GRAFMÜLLER


*Abstract*—**The electromagnetic field in a conical waveguide with an elliptical cross section is calculated with the aid of two scalar potentials which satisfy the Helmholtz equation, the Dirichlet, and the Neumann boundary condition, respectively. The transverse parts of the solutions of the Helmholtz equation in the sphero-conal coordinate system are products of periodic and nonperiodic Lamé functions. These functions allow a mode definition similar to that for conventional waveguides. Some transverse modal field distributions, together with the corresponding Lamé functions, are graphically represented for a special elliptic conical waveguide.**


## I. Introduction

The electromagnetic field in the interior of a cone with an elliptical cross section can be built up by solutions of the Helmholtz equation in a similar manner as is done in the case of rectangular or circular waveguides [1], [2]. For these calculations, the sphero-conal coordinate system is used which has elliptic cones as coordinate surfaces.


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The authors are with Lehrstuhl für Theoretische Elektrotechnik, Ruhr-Universität Bochum, 4630 Bochum, W. Germany.


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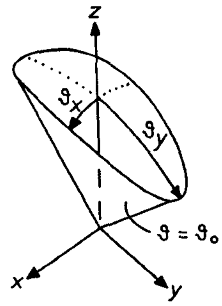


Fig. 1. Geometry of a cone with elliptic cross section.

The resulting modes show field configurations similar to those of modes in elliptic hollow pipes described by Chu [3]. Field lines of the lowest mode have already been given by Ng [4], but higher order modes have not been calculated as far as the authors know.

In this paper, only a short survey of the solution theory of the Helmholtz equation in sphero-conal coordinates and the involved Lamé functions is given. Details may be found in [4]–[11].

## II. Solution of Maxwell's Equations in Sphero-conal Coordinates

The relation between Cartesian coordinates and the sphero-conal coordinates $r, \vartheta, \varphi$ can be defined by (1). In the special case $k^2 = 1$, these coordinates become the well-known spherical coordinates, with the $z$-axis being the polar axis

$$\begin{aligned} x &= r \sin\vartheta \cos\varphi \\ y &= r\sqrt{1-k^2\cos^2\vartheta}\,\sin\varphi \\ z &= r\cos\vartheta\sqrt{1-k'^2\sin^2\varphi} \\ &0 \le k,\ k' \le 1, \quad k^2 + k'^2 = 1 \\ &0 \le r < \infty, \quad 0 \le \vartheta \le \pi, \quad 0 \le \varphi \le 2\pi. \end{aligned} \tag{1}$$

The coordinates surfaces $\vartheta = \vartheta_0 = \text{const.}$ are cones with an elliptic cross section (Fig. 1). The extreme flare angles are

$$\vartheta_x = \vartheta_0$$

and

$$\vartheta_y = \arccos(k \cdot \cos\vartheta_0) \quad (\vartheta_y \ge \vartheta_x \text{ if } \vartheta_0 \le \pi/2). \tag{2}$$

The electromagnetic field in such a cone can be calculated with the aid of the substitution

$$\begin{aligned} \vec{H} &= \text{curl}(\psi^E \vec{r}) \quad &&\text{for TM-waves and} \\ \vec{E} &= -\text{curl}(\psi^H \vec{r}) \quad &&\text{for TE-waves, respectively.} \end{aligned} \tag{3}$$

Then Maxwell's equations demand that the scalar functions $\psi^E$ and $\psi^H$ must satisfy the Helmholtz equation

$$\Delta\psi^{E,H} + \kappa^2\psi^{E,H} = 0 \quad (\kappa\text{: wave number}). \tag{4}$$

In detail, (3) reads for TM-waves

$$\begin{aligned} E_r &= \frac{1}{j\omega\epsilon_0}\left[\frac{\partial^2(r\psi^E)}{\partial r^2} + \kappa^2 r\psi^E\right], \quad &H_r &= 0 \\ E_\vartheta &= \frac{1}{j\omega\epsilon_0 h_\vartheta}\frac{\partial^2(r\psi^E)}{\partial r\partial\vartheta}, \quad &H_\vartheta &= \frac{r}{h_\varphi}\frac{\partial\psi^E}{\partial\varphi} \\ E_\varphi &= \frac{1}{j\omega\epsilon_0 h_\varphi}\frac{\partial^2(r\psi^E)}{\partial r\partial\varphi}, \quad &H_\varphi &= -\frac{r}{h_\vartheta}\frac{\partial\psi^E}{\partial\vartheta} \end{aligned} \tag{5}$$

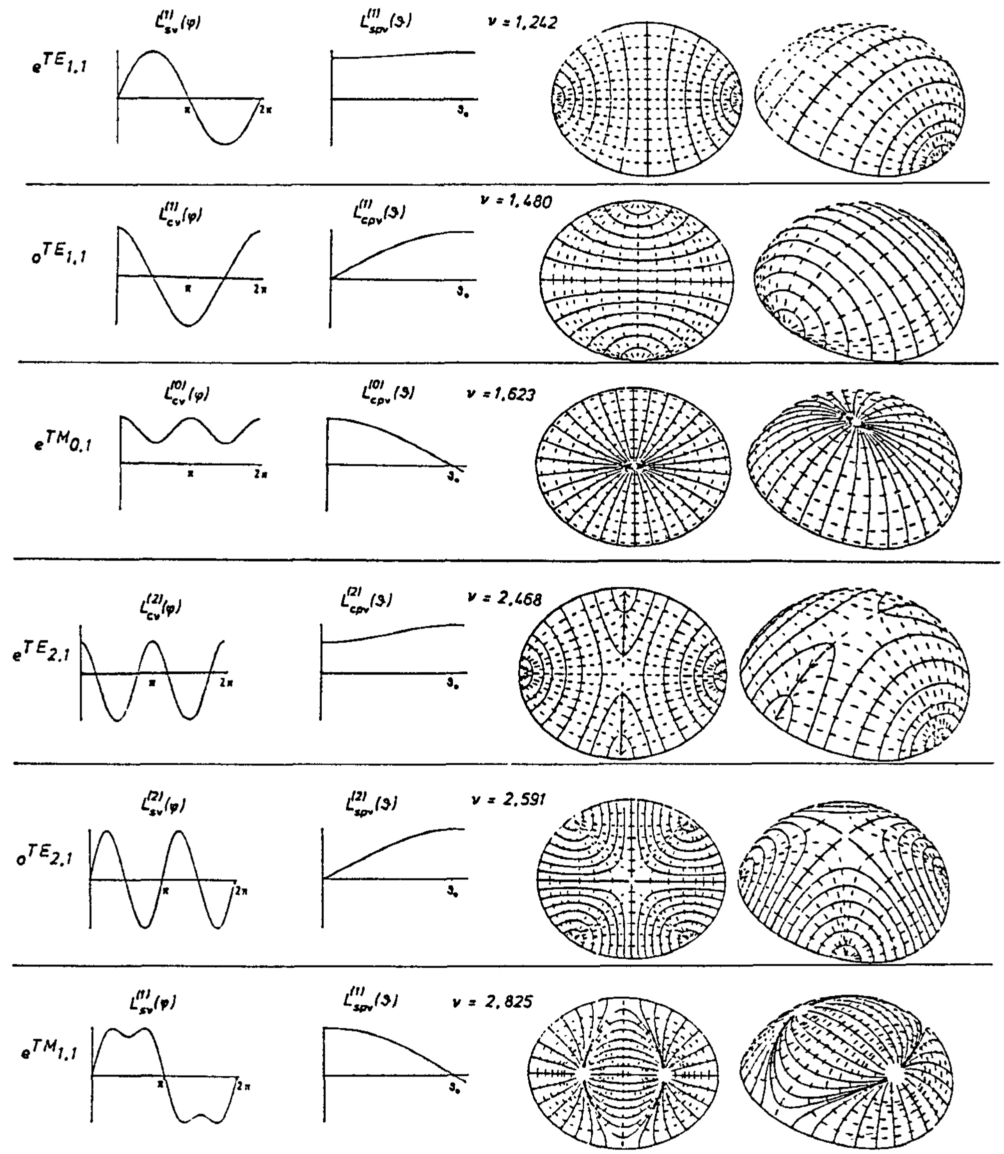


Fig. 2. Transverse modal field distributions for a conical waveguide with elliptic cross section ($\vartheta_x = 60°$, $\vartheta_y = 70°$, $k^2 = 0.468$. —— electric lines, ---- magnetic lines.

and for TE-waves

$$H_r = \frac{1}{j\omega\mu_0}\left[\frac{\partial^2(r\psi^H)}{\partial r^2} + \kappa^2 r\psi^H\right], \quad E_r = 0$$

$$H_\vartheta = \frac{1}{j\omega\mu_0 h_\vartheta}\frac{\partial^2(r\psi^H)}{\partial r\partial\vartheta}, \qquad E_\vartheta = -\frac{r}{h_\varphi}\frac{\partial\psi^H}{\partial\varphi}$$

$$H_\varphi = \frac{1}{j\omega\mu_0 h_\varphi}\frac{\partial^2(r\psi^H)}{\partial r\partial\varphi}, \qquad E_\varphi = \frac{r}{h_\vartheta}\frac{\partial\psi^H}{\partial\vartheta}. \tag{6}$$

The metric scale factors $h_\vartheta$ and $h_\varphi$ are given by

$$h_\vartheta = \left|\frac{\partial\vec{r}}{\partial\vartheta}\right| = r\cdot\sqrt{\frac{k^2\sin^2\vartheta + k'^2\cos^2\varphi}{1-k^2\cos^2\vartheta}}$$

$$h_\varphi = \left|\frac{\partial\vec{r}}{\partial\varphi}\right| = r\cdot\sqrt{\frac{k^2\sin^2\vartheta + k'^2\cos^2\varphi}{1-k'^2\sin^2\varphi}}. \tag{7}$$

Equation (4) can be solved in sphero-conal coordinates by separation of variables $\psi(r,\vartheta,\varphi) = R_\nu(r)\cdot\theta_{\nu,\lambda}(\vartheta)\cdot\phi_{\nu,\lambda}(\varphi)$, with separation constants $\nu$ and $\lambda$. This leads to three ordinary differential equations

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[\kappa^2r^2 - \nu(\nu+1)\right]R = 0$$

$$\sqrt{1-k^2\cos^2\vartheta}\,\frac{d}{d\vartheta}\left\{\sqrt{1-k^2\cos^2\vartheta}\,\frac{d\theta}{d\vartheta}\right\} + \left[\nu(\nu+1)(1-k^2\cos^2\vartheta) - \lambda\right]\theta = 0$$

$$\sqrt{1-k'^2\sin^2\varphi}\,\frac{d}{d\varphi}\left\{\sqrt{1-k'^2\sin^2\varphi}\,\frac{d\phi}{d\varphi}\right\} + \left[\lambda - \nu(\nu+1)k'^2\sin^2\varphi\right]\phi = 0. \tag{8}$$

The one for $R_\nu(r)$ is the equation of the spherical cylinder functions of order $\nu$ ($j_\nu(\kappa r), n_\nu(\kappa r), h_\nu^{(1)}(\kappa r)$ or $h_\nu^{(2)}(\kappa r)$). The other two are the so-called Lamé equations which are coupled with each other by separation constants $\nu$ and $\lambda$. From the geometry of the problem follows that the functions $\phi_{\nu,\lambda}(\varphi)$ must be periodic — called periodic Lamé functions — but the $\Theta_{\nu,\lambda}(\vartheta)$ are general nonperiodic Lamé functions.

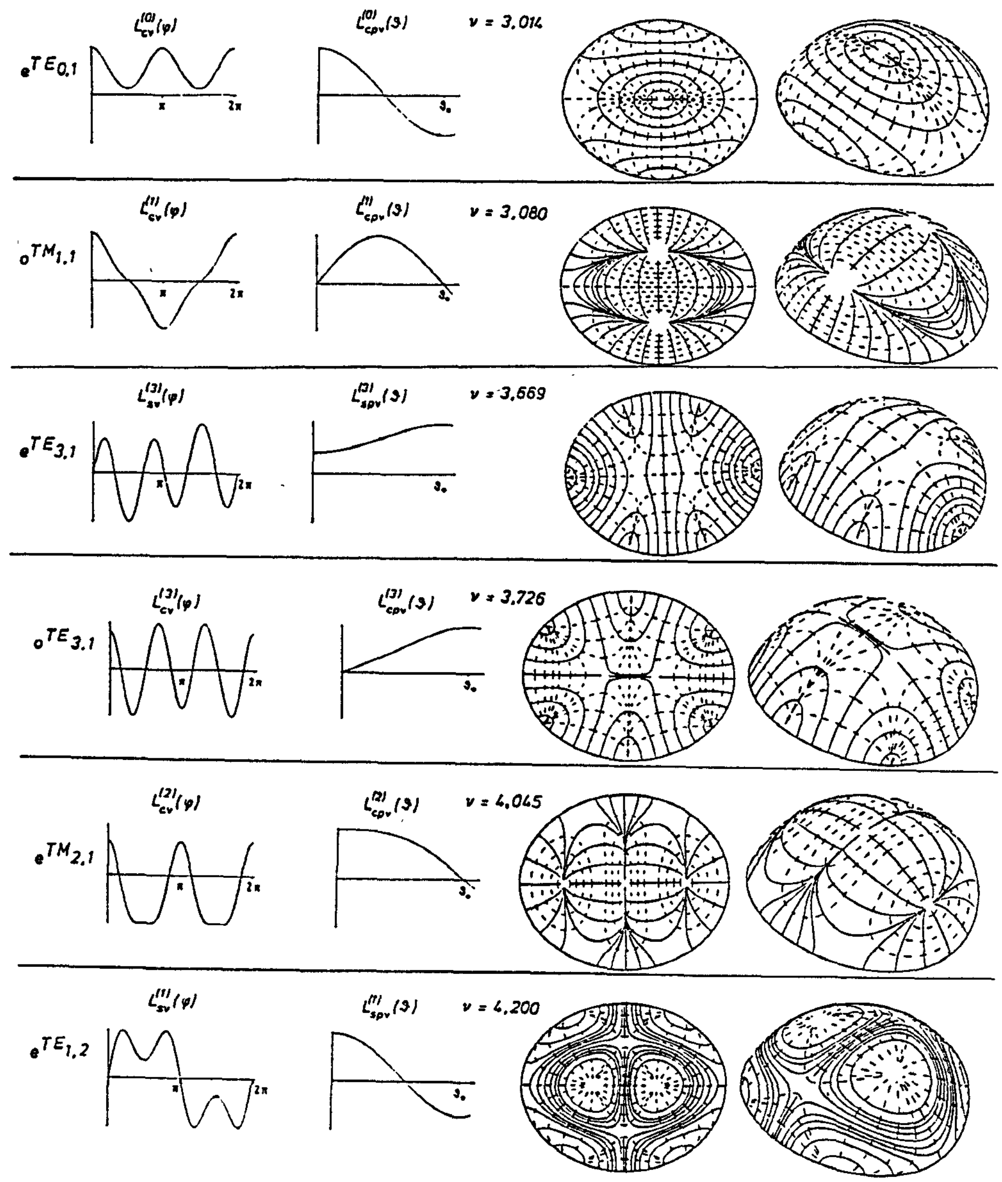


Fig. 3. Transverse modal field distributions for a conical waveguide with elliptic cross section. ($\vartheta_x = 60°$, $\vartheta_y = 70°$, $k^2 = 0.468$). —— electric lines, ---- magnetic lines.

The periodic Lamé functions can be classified into four types which are distinguished by their symmetry relative to $\varphi = \pi/2$ and by their periodicity. The functions are calculated with the aid of Fourier series [11]. The four types are:[1]

1) even symmetric and $\pi$-periodic; denoted as $L_{cv}^{(2\mu)}(\varphi)$ $(\mu = 0,1,\cdots)$
2) odd symmetric and $2\pi$-periodic;[2] denoted as $L_{cv}^{(2\mu+1)}(\varphi)$ $(\mu = 0,1,\cdots)$
3) odd symmetric and $\pi$-periodic; denoted as $L_{sv}^{(2\mu)}(\varphi)$ $(\mu = 1,2,\cdots)$
4) even symmetric and $2\pi$-periodic;[2] denoted as $L_{sv}^{(2\mu+1)}(\varphi)$ $(\mu = 0,1,\cdots)$.

The upper index in this notation is the number of zeros in the interval $[0,\pi)$ and is related to the separation constant $\lambda$. For each function type this constant $\lambda$ can be calculated from the demanded symmetry and periodicity as a function of the other constant $\nu$.

To each of the periodic Lamé functions corresponds exactly one nonperiodic Lamé function yielding a definite field distribution. The nonperiodic functions can be represented by a sum over the associated Legendre functions, the lower index of the latter being $\nu$ and the upper index being the summation index. The nonperiodic Lamé functions are denoted as

1) $L_{cp\nu}^{(2\mu)}(\vartheta)$
2) $L_{cp\nu}^{(2\mu+1)}(\vartheta)$
3) $L_{sp\nu}^{(2\mu)}(\vartheta)$
4) $L_{sp\nu}^{(2\mu+1)}(\vartheta)$.

The separation constant $\nu$ has to be chosen in such a way that the boundary condition $\vec{E}_{\tan} = 0$ is satisfied at the surface $\vartheta = \vartheta_0$ of the perfectly conducting cone. This requirement is equivalent to the transcendental equations $\Theta_{\nu,\lambda}(\vartheta_0) = 0$ for TM-waves and

$$\left.\frac{d}{d\vartheta}\Theta_{\nu,\lambda}\right|_{\vartheta_0} = 0 \quad \text{for TE-waves} \tag{9}$$

[1] This classification was suggested by Ince [6].
[2] $2\pi$-periodic means not $\pi$— but $2\pi$-periodic.

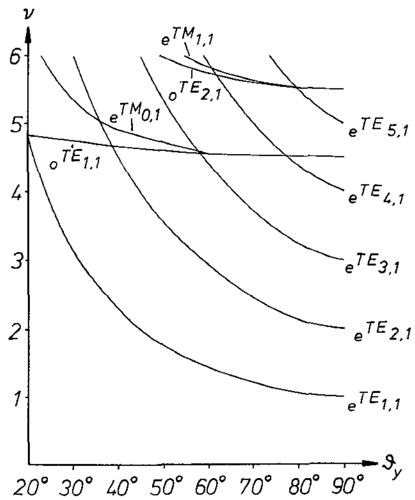


Fig. 4. Separation constant $\nu$ as a function of $\vartheta_y$ ($\vartheta_x = 20°$).

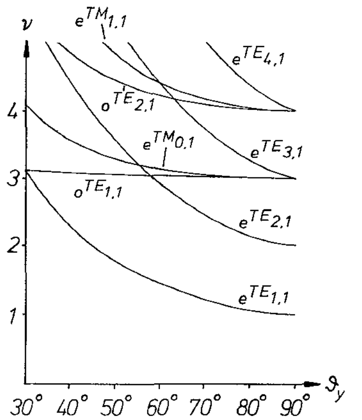


Fig. 5. Separation constant $\nu$ as a function of $\vartheta_y$ ($\vartheta_x = 30°$).

which have been solved numerically to get the unknown values $\nu$. The theory of periodic and nonperiodic Lamé functions and the determination of the separation constants are outlined in [10]–[12].

## III. Modes in Elliptic Conical Waveguides

The properties of the products $\Theta_{\nu,\lambda}(\vartheta)\cdot\phi_{\nu,\lambda}(\varphi)$ allow a mode definition similar to that usually chosen for the modes in a circular waveguide [1]. There the indices $m$ and $n$ of a mode are given by the solutions ${}^{\sin}_{\cos}(m\varphi)\cdot J_m(\alpha_n r)$ with $\alpha_n$ being the $n$th root of $J_m(\alpha a) = 0$ for TM-modes, and $J'_m(\alpha a) = 0$ for TE-modes ($a$: radius of the circular waveguide), respectively.

To generalize this definition one can interpret the index $m$ as the number of zeros of ${}^{\sin}_{\cos}(m\varphi)$ in the interval $[0, \pi)$. This interpretation also holds for the elliptic cone, so that the index $m$ of a mode is the upper index of the periodic Lamé functions and is related to the separation constant $\lambda$. To make this relation definite one makes a distinction between even and odd modes (abbreviated by $e$ and $o$, respectively) depending on the symmetry of the periodic Lamé function. Referred to (9) the index $n$ must be interpreted in the same way as in the case of circular waveguides.

Figs. 2 and 3 show the first 12 modes in an elliptic cone with flare angles $\vartheta_x = 60°$ and $\vartheta_y = 70°$ ($k^2 = 0.468$) arranged with increasing $\nu$. To the left of the figures the corresponding periodic and nonperiodic Lamé functions are sketched, and to the right two projections of the field lines which run on a sphere are shown.

The field configuration of the modes is similar to that of the modes in elliptic waveguides calculated by Chu [3] with the exception of the ${}_e\mathrm{TM}_{0,1}$-mode whose field distribution in our case is similar to that of a circular waveguide.

The field configuration of some modes depends on the eccentricity of the elliptic cone. So the field configuration of the ${}_e\mathrm{TM}_{2,1}$-mode shown is not like that expected from circular waveguide [2], because the Lamé functions $L_{c\nu}^{(2)}(\varphi)$ and $L_{cp\nu}^{(2)}(\vartheta)$ differ principally from functions in that case. With increasing $k^2$ this appearance vanishes.

The order of the modes also depends on geometry. This is shown in Figs. 4 and 5.

## IV. Conclusions

The electromagnetic field in a conical waveguide with an elliptic cross section is described with the aid of two scalar potentials which satisfy the Helmholtz equation, the Dirichlet, and the Neumann boundary condition, respectively. The behavior of the transverse parts of the solutions of the Helmholtz equation allows a generalization of the mode definition used for conventional waveguides.

The field configuration of some modes depends on the eccentricity of the elliptic cone. Also, the order of the modes depends on geometry.

For an elliptic cone with flare angles $\vartheta_x = 60°$ and $\vartheta_y = 70°$ the first 12 modes are graphically represented by the transverse modal field distributions together with the corresponding periodic and nonperiodic Lamé functions.

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